March 2, 2017

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Look at all possible topologies on $P = \prod_{\alpha \in I} X_\alpha$

$$\{\phi, P\} \subset \cdots\cdots \subset \underbrace{\mathcal{T}_\Pi \subset \cdots \subset \mathcal{T}_{Box} \subset \cdots\cdots \subset \mathcal{P}(P)}$$

guarantee continuity of each projection mapping $\pi_\beta : P \longrightarrow X_\beta$

Another usual need for study

$$W \xrightarrow{f} P = \prod_{\alpha \in I} X_\alpha \xrightarrow{\pi_\beta} X_\beta$$

f is continuous $\Leftrightarrow$ each $\pi_\beta \circ f$ is so.

"$\Rightarrow$" is a natural property

"$\Leftarrow$" helps us all the time, e.g.

$$(x,y) \longmapsto (xy\sin(x+y),\ (x+y)e^{x-y},\ x^2-y^2)$$

To check continuity, one only need to verify each coordinate function.

Now, want to have the property that

$$(W, \mathcal{T}_W) \xrightarrow{f} (P, \mathcal{T}) \xrightarrow{\pi_\beta} (X_\beta, \mathcal{T}_\beta)$$

if each $\pi_\beta \circ f$ is continuous then $\underbrace{\text{so is } f}_{f^{-1}(V) \in \mathcal{T}_W}$,

where $\mathcal{T}$ is chosen from

$$\{\phi, P\} \subset \cdots \subset \mathcal{T}_\Pi \subset \cdots \subset \mathcal{T}_{Box} \subset \cdots\cdots \subset \mathcal{P}(P)$$

easy

difficult

Theorem. $\mathcal{T}_\Pi$ is the maximal topology for $P = \prod_{\alpha \in I} X_\alpha$ such that for all $f : W \to P$, it is continuous whenever each $\pi_\beta \circ f : W \to X_\beta$ is continuous.

Proof. To prove maximality, let $\mathcal{T}$ be a topology for P doing the job.

Try to show $\mathcal{T} \subset \mathcal{T}_\Pi$.

Take any $U \in \mathcal{T}$, why does $U \in \mathcal{T}_\Pi$?

Simply consider $W = P$, $\mathcal{T}_W = \mathcal{T}_\Pi$, $f = id$.

i.e., $id : (P, \mathcal{T}_\Pi) \to (P, \mathcal{T})$

Clearly, each $\pi_\beta \circ id = \pi_\beta : (P, \mathcal{T}_\Pi) \to X_\beta$ is continuous, so is $id : (P, \mathcal{T}_\Pi) \to (P, \mathcal{T})$

Hence, $id^{-1}(U) = U \in \mathcal{T}_\Pi$.

We still need to show $\mathcal{T}_\Pi$ does the job.

Assume that $f \circ \pi_\beta : (P, \mathcal{T}_\Pi) \to X_\beta$ is continuous and $U \in \mathcal{T}_\Pi$. Try to prove $f^{-1}(U) \in \mathcal{T}_W$.

As $U = \bigcup \bigcap_{k=1}^{n} \pi_{\beta_k}^{-1}(V_k)$, $V_k \in \mathcal{T}_{\beta_k}$

$$f^{-1}(U) = \bigcup \bigcap_{k=1}^{n} f^{-1} \pi_{\beta_k}^{-1}(V_k)$$

$$= \bigcup \bigcap_{k=1}^{n} (\pi_{\beta_k} \circ f)^{-1}(V_k) \in \mathcal{T}_W$$

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Example. Let $I = \mathbb{N}$, $(X_k, \mathcal{T}_k) = (\mathbb{R}, \text{std})$ $\forall k \in I$

$$(W, \mathcal{T}_W) = (\mathbb{R}, \text{std})$$

$$f : (\mathbb{R}, \text{std}) \longrightarrow \mathbb{R}^{\mathbb{N}} = \prod_{k \in \mathbb{N}} \mathbb{R}$$

$$t \longmapsto (t, t, t, t, \cdots, t, \cdots)$$

First, on $(\mathbb{R}^{\mathbb{N}}, \mathcal{T}_{\pi})$,

each $\pi_k \circ f : t \longmapsto t : (\mathbb{R}, \text{std}) \longrightarrow (\mathbb{R}, \text{std})$

is continuous obviously.

Thus, $f : (\mathbb{R}, \text{std}) \longrightarrow (\mathbb{R}^{\mathbb{N}}, \mathcal{T}_{\pi})$ is so.

However, $f : (\mathbb{R}, \text{std}) \longrightarrow (\mathbb{R}^{\mathbb{N}}, \mathcal{T}_{Box})$ is not!

Consider $V \in \mathcal{T}_{Box}$

$$V = (-1,1) \times \left(\tfrac{-1}{2}, \tfrac{1}{2}\right) \times \left(\tfrac{-1}{3}, \tfrac{1}{3}\right) \times \cdots \times \left(\tfrac{-1}{n}, \tfrac{1}{n}\right) \times \cdots \times \cdots$$

$$\ni (0, 0, 0, 0, 0, \cdots, 0, 0, \cdots, 0, \cdots)$$

$f^{-1}(V) \subset \mathbb{R}$ contains $0 \in \mathbb{R}$ but it cannot be open. Otherwise, $\exists\ \varepsilon > 0$

$$(-\varepsilon, \varepsilon) \subset f^{-1}(V)$$

But $f(-\varepsilon, \varepsilon) \subset V$

$$f(-\varepsilon, \varepsilon) = (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \times \cdots \times (-\varepsilon, \varepsilon) \times \cdots$$

contradiction